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LETTER TO THE EDITOR

Critical behaviour of anisotropic 'superelastic' central-force percolation

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Abstract. We study central-force percolation on a triangular lattice where the bonds are springs that can freely rotate around the nodes. All bonds in one preferred direction are infinitely rigid. In the other two directions a proportion p of the springs chosen at random are infinitely rigid while the rest have a finite strength. For this specific problem, the threshold at which the network becomes infinitely rigid corresponds to the usual bond-percolation threshold of a square lattice and is therefore exactly equal to 0.5. We use a transfer-matrix algorithm to study the elastic modulus of strips of width ranging from 2 to 32 and length 10^5. We obtain an estimate of the critical exponent relative to the scaling of the elastic moduli with the strip width, 0.99 ± 0.03, close to s/ν for the corresponding isotropic electric problem.

Elastic properties of randomly depleted media (and lattices) are different from other transport properties even though they are closely related to them both in nature and formalism. For instance, in usual percolation, the elastic critical exponent is very different from the conductivity one [1] even though they might be related [2]. In central-force percolation [3] the difference between the elastic and other transport properties is even greater; here only the elastic behaviour is critical [4].

Two-dimensional percolation is already a problem for which many exact results are known [5]. This concerns either some thresholds for regular lattices (bond percolation on a square lattice, site percolation on a triangular lattice, etc) or even some critical exponents: most static ones are known and some relations among transport exponents have been proposed [2, 6]. Conversely, no exact relations are known on the problem of central-force percolation. In particular, the lack of knowledge of the precise value of the threshold on any lattice renders very difficult the task of evaluating the critical exponents.

In this letter we present a study of *anisotropic* 'superelastic' central-force percolation on a triangular lattice. This is the anisotropic analogue of the dual problem to central-force percolation, namely the 'superelastic' problem, where the bonds of the isotropic network either are infinitely rigid with probability p or have finite strength with probability $1-p$, as first studied by Sahimi and Goddard [7]. The bonds are Hookean linear springs free to rotate at their junction nodes. The anisotropy is introduced by letting *all* bonds in a preferred direction be infinitely rigid and a certain proportion p of randomly distributed bonds in the other two directions are also infinitely rigid while the other bonds have a finite strength. The anisotropy we introduce simplifies

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the problem in a way that allows one to derive *exactly* the same rigidity threshold as for usual bond percolation on a square lattice. Thus we have $p_c = 0.5$.

Such a simplification is not new: the introduction of a similar anisotropy in directed percolation on a square lattice, all bonds being present in one direction, makes it possible to map this problem exactly onto a solvable random walk problem [8]. However, the price of this simplification is that the universality class that this problem belongs to is different from the universality class of the isotropic problem. As a result, the critical exponents in this anisotropic directed percolation problem are different from those of the isotropic case. For instance, $\nu_{\parallel} = 2$ in the anisotropic case as compared to $\nu_{\parallel} = 1.7334$ for two-dimensional isotropic directed percolation [9].

Similarly, we cannot expect the results we present here to be valid for the isotropic 'superelastic' central-force problem. However, they give an insight into the relationship between this elastic problem and the usual percolation problem.

The difficulty of central-force percolation lies in the fact that the nature of the field to be transported (or not!) by the structure is essentially a vector one. A consequence of this is the non-locality of the transport [3]. For instance, the equivalent of the 'red bonds' or 'singly connected bonds' in usual percolation [10] has very different characteristics in this problem. One could define them in the central-force problem as those bonds that would ruin the rigidity of the structure if they were cut. With this definition, one can exhibit clusters of all sizes in which all bonds are 'red' [11] (see also [4]). The shape of these structures is far from being one dimensional as they would be in the usual percolation problem. However, this non-locality associated with the vector nature of central-force percolation is 'dampened' by the introduction of the anisotropy described above.

We now show that the rigidity threshold is 0.5 for the anisotropic central-force problem on a triangular lattice. This is then also the threshold for the 'superelastic' problem that we are interested in here. In our triangular lattice, we can separate the set of parallel lines where all bonds are present (let us say direction 1). The rest of the network then consists of an oblique square lattice. On this square lattice the proportion of present bonds is p. Let us first consider the case where p is smaller than the bond-percolation threshold for the square lattice, $p_c = 0.5$. In this case there exist only finite clusters. The whole triangular lattice cannot resist any force which is not parallel to direction 1. This is illustrated in figure 1. If a force $\boldsymbol{F}$ is applied onto site A, we consider the cluster of sites connected to A on the square sublattice. We can draw an imaginary curve C that surrounds this cluster and that will not cross any bond that is not parallel to direction 1. In a central-force model, a bond can only support a force that is parallel to itself, because of the free rotation of the springs at the nodes. Now, writing down the equilibrium equation for the domain surrounded by C, we see that we have to equilibrate $\boldsymbol{F}$ by the sum of forces carried by the bonds that cross C and are thus parallel to direction 1. Therefore $\boldsymbol{F}$ must be parallel to direction 1, i.e. the structure is rigid only in this direction. Hence p_c is a lower bound on the rigidity threshold.

When p is larger than p_c and point A, on which the force $\boldsymbol{F}$ is exerted, belongs to the infinite cluster of the square sublattice, we can extract a continuous path from A to any other point B on this infinite cluster arbitrarily far from A. This path plus all lines in direction 1 that are crossed by the path form a structure that can transmit the force $\boldsymbol{F}$ from A to B, whatever its direction might be. It is straightforward to find the distribution of force in such a substructure. The corresponding stress field is an admissible field for the entire structure and this implies that it can transmit force

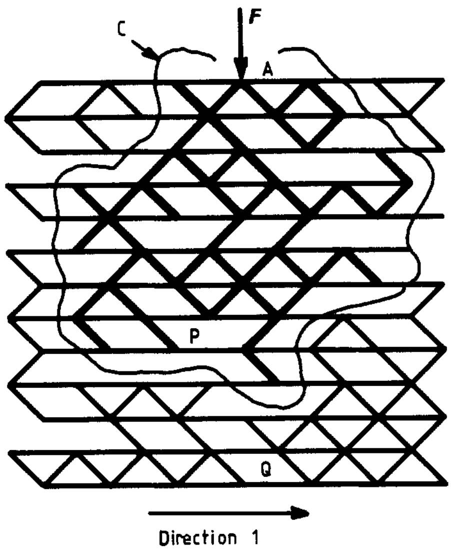

Figure 1. Geometry of the lattice for the anisotropic central-force percolation problem. When p is smaller than the usual bond-percolation threshold, the structure cannot hold any force that is not along the horizontal direction 1 where all bonds are present. We can transmit a force from A to P, but not from A to Q.

pointing in any direction over an arbitrary distance. This is the natural criterion we choose to determine whether the lattice is rigid or not. Thus, since an infinite cluster, which by the above argument is rigid, is present when p is larger than p_c, this threshold is also an upper bound for the rigidity threshold.

This leads to the rigidity threshold being equal to $p_c = 0.5$. To compare this problem to the anisotropic 'superelastic' problem, let the elastic springs become infinitely rigid, and the missing (or infinitely soft) springs acquire a finite elasticity. A rigid structure in the previous problem will now be an *infinitely* rigid structure. Now the appearance of an infinitely rigid structure of infinite size constitutes the rigidity transition. Since the geometry is unchanged, the rigidity transition will occur at $p = 0.5$ also in this problem.

We studied the anisotropic 'superelastic' problem through finite-size scaling analysis thanks to a transfer-matrix algorithm [12]. Usual percolation, and particularly in this context conductivity, has been dealt with intensively [13]. It turns out that the superconductor–conductor percolation problem (corresponding to the 'superelastic' problem we study here) in the transfer-matrix approach is much less affected by correction-to-scaling terms than the usual conductor–insulator problem [14]. The main reason for this lies in the fact that the former problem can be treated with periodic boundary conditions efficiently so as to reduce edge effects, in contrast to the latter where this is useless.

We generated strips of length 10^5 of width ranging from 2 to 32 at the rigidity threshold $p_c = 0.5$. We used periodic boundary conditions in the transverse direction. One end of the strip was fixed and at the other end a force was imposed. The elastic moduli for both shear and compression were then calculated. Shear corresponds to applying the force in the transverse direction and compression to applying the force along the direction along the strip.

We first studied the geometry of figure 2(a), i.e. the preferred direction is along the width of the strip. These lines thus form closed loops. (The boundary conditions are easy to visualise as giving the strip a cylindrical shape.) The evolution of the compliance (i.e. the inverse of the elastic modulus) with the width of the strip is shown in figure 2(b). The compression compliance decreases very slowly, whereas the shear compliance decreases exponentially. One should again note the correspondence between the 'superelastic' and the usual (diluted) central-force problem. If the shear compliance in the 'superelastic' problem decreases exponentially, then the corresponding elastic modulus *increases* at the same rate. This means that the shear elastic modulus for the usual, or diluted, problem will decrease slowly or tend towards a constant. Conversely, if in the 'superelastic' problem, the compression compliance decreases slowly, then in the diluted problem the compression modulus will decrease exponentially.

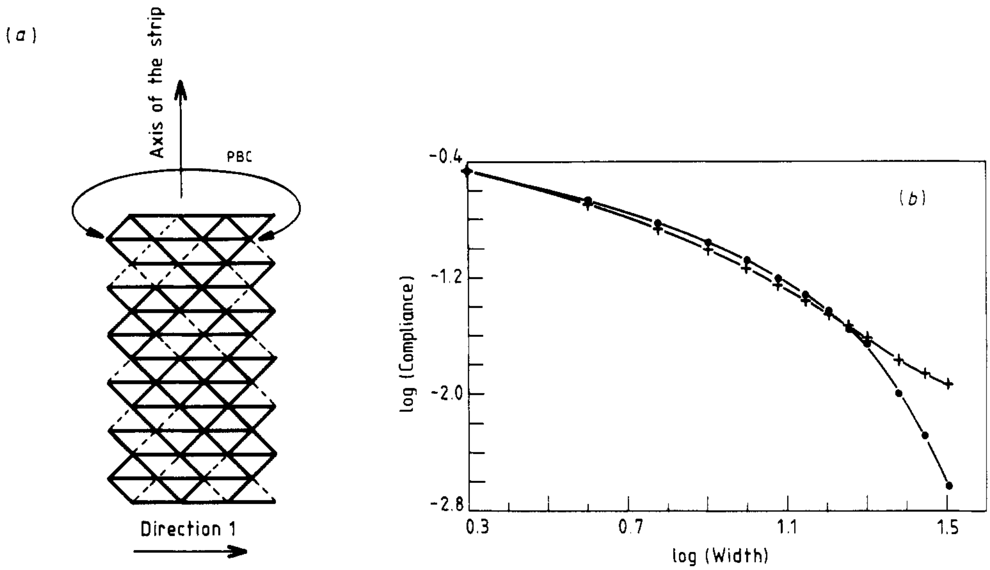

Figure 2. (a) Geometry of the lattice strip used for the computation of the compliance reported in (b). The broken lines correspond to normal springs whereas the full lines indicate infinitely rigid bonds and PBC denotes the periodic boundary conditions in the transverse direction. (b) Log_{10} of the elastic compliances for shear (●) and compression (+) plotted against $\log_{10}$ of the width of the strip for $p = 0.5$ in the geometry of (a).

This seems to contradict the claim that the rigidity threshold is at $p = 0.5$. In fact, this unexpected behaviour is a result of the boundary conditions we chose for the direction in which all bonds are infinitely rigid (they form loops). This is very different from having fixed boundaries, since these infinitely rigid rings may rotate in the direction transverse to the length of the strip. Thus, structures of the type displayed in figure 3 will be rigid if the force applied is a shear, and will be 'soft' if the force applied is a compression. These structures, which are peculiar to the boundary conditions we have chosen, occur frequently enough to destroy the scaling behaviour of the elastic moduli.

To avoid these configurations, we also studied the case where the preferred direction forms a $\pi/6$ angle with the axis of the strip, still for the 'superelastic' problem. This geometry is shown in figure 4(a) and the corresponding results for the compliances

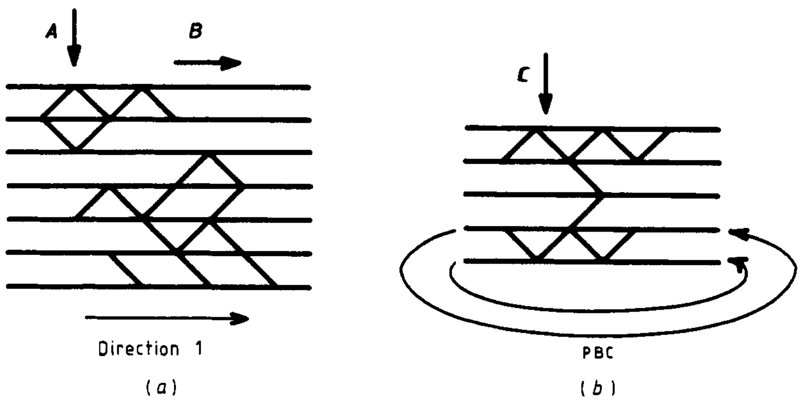

Figure 3. (*a*) A configuration that cannot transmit any force $\boldsymbol{A}$ perpendicular to direction 1, but acts as a rigid structure if a force $\boldsymbol{B}$ is applied. (*b*) We see that even though the structure is connected on the square sublattice from top to bottom, it cannot transmit a force $\boldsymbol{C}$ as a result of the periodic boundary conditions (PBC). These configurations are responsible for the unexpected behaviour in the corresponding 'superelastic' problem seen in figure 2(*b*).

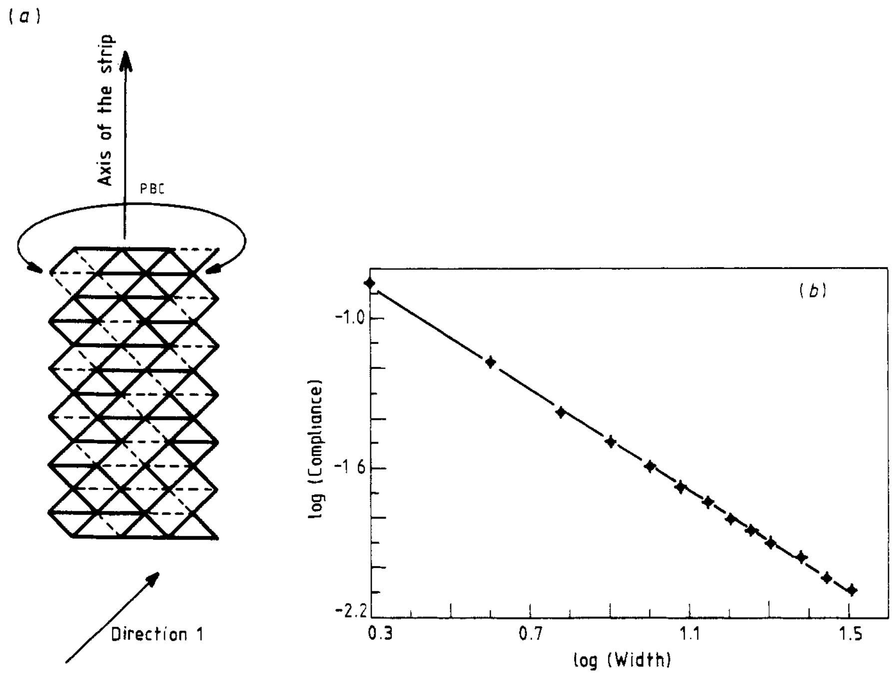

Figure 4. (*a*) Geometry of the lattice strip used for the computation of the compliance reported in (*b*). The broken lines correspond to normal springs whereas the full lines indicate infinitely rigid bonds and PBC denotes the periodic boundary conditions. (*b*) Log_{10} of the elastic compliances for shear (●) and compression (+) plotted against $\log_{10}$ of the width of the strip at $p=0.5$ in the geometry of (*a*). The data points for the two compliances are indistinguishable. The slope of the best-fit line has an asymptotic value of $-g/\nu=0.99\pm0.03$.

are shown in figure 4(*b*). These data indicate that for both shear and compression, the compliance, C, varies with the width w to some power x, $C \sim w^{-x}$, where $x = 0.99 \pm 0.03$, as determined by a least-squares fit. Thanks to finite-size scaling we would expect to obtain $x = g/\nu$ where g is the critical exponent describing the divergence of the elastic modulus at the threshold, $E \sim (p_c - p)^{-g}$, and ν describes the divergence of the correlation length, $\xi \sim |p_c - p|^{-\nu}$. The geometric critical behaviour of this problem should be identical to that of the square sublattice which is at the percolation threshold. Therefore ν should be $\frac{4}{3}$ [15] and the value of g/ν is very close to the one obtained for the ratio s/ν where s is the conductivity exponent for superconductor-conductor percolation. We should also note that this exponent is close to the one describing the divergence of the elastic modulus for superrigid-rigid percolation *with bond-bending*, as it has been suggested that the latter exponent is equal to s [6].

The problem of anisotropic central-force percolation that we have introduced here can be encountered in some practical cases such as in the piling of cylinders of slightly fluctuating radii [16]. However, in this context of piling, when the asymmetry of the contact between the cylinders, i.e. that the contact can resist compression but not traction (extension), is taken into account, then the relevant threshold is no longer the isotropic one but rather the *directed* percolation threshold which for the square bond-percolation problem is 0.644 701 [9].

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